Modelling power-law distributed interevent times

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Many human-related activities show power-law decaying interevent time distribution with exponents usually varying between 1 and 2. We study a simple task-queuing model which can reproduce this property and we show exact results for the asymptotic behaviour of the model. The model satisfies a scaling law between the exponents of interevent time distribution (β) and autocorrelation function (α): $\alpha+\beta=2$. This law is general for renewal processes with power-law decaying interevent time distribution. We conclude that slowly decaying autocorrelation function indicates long-range dependency only if the scaling law is violated.

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Studying human activity patterns is of central interest due to the wide practical usage. Understanding the dynamics underlying the timing of various human activities – such as starting a phone call, sending an e-mail or visiting a web-site – are crucial to modelling the spreading of information or viruses [1]. Modelling human dynamics is also important in resource allocation. It has been shown that for many human activities the interevent time distribution follows a power law with exponents usually varying between 1 and 2. Processes with power-law decaying interevent time distribution look very different from the Poisson process which has been used to model the timing of human activities before [2, 3]. While time series from the latter look rather homogeneous, the former processes produce bursts of rapidly occurring events which are separated by long inactive periods.

Some examples where power-law decaying interevent time distribution has been observed are email-communication (with exponent $\beta\approx 1$, [4]), surface mail communication ($\beta\approx 1.5$, [5]), web-browsing ($\beta\approx 1.2$, [6]) and library loans ($\beta\approx 1$, [7]). In some other cases a monotonous relation has been reported between the user activity and the interevent time exponent, for example in short-message communication ($\beta\in(1.5,2.1)$, [8]) or in watching movies ($\beta\in(1.5,2.7)$, [9]). In a recent paper there can be found a distribution of exponents of various channels of communications [10]. These observations make it necessary to find a model in which the interevent time exponent is tunable.

Similar bursty behaviour has been observed in other natural phenomena, for example in neuron-firing sequences [11] or in the timings of earthquakes [12]. The interevent time distribution does not give us any information about the dependency among the consecutive actions. Correlation between events is usually characterised by the autocorrelation function of the timings of detected events. Bursty behaviour is often accompanied by power-law decaying autocorrelation function [13] which is usually thought to indicate long-range dependency, see e.g. [14]. However, time series with independent power-law distributed interevent times show long-range correlations [15, 16].

This letter is organized as follows. We start with introducing a task-queuing model which has an advantage compared to the Barabási-model [17], namely that the observable is not the waiting time of an action (between adding to the list and executing it) but the interevent time between similar activities. We determine the asymptotic decay of the interevent time distribution in a simple limit of the model. We give a simple proof of the scaling law between the exponents of the interevent time distribution and the autocorrelation function based on Tauberian theorems. Finally, we demonstrate that the scaling law can be violated if the interevent times are long-range dependent.

The model: We assume that people have a task list of size N and they choose their forthcoming activity from this list. The list is ordered in the sense that the probability w_i of choosing the i^{th} activity from the list is monotonically decreasing as a function of position i. The chosen activity is going to be executed and it jumps to the front of the list (fig.1). This

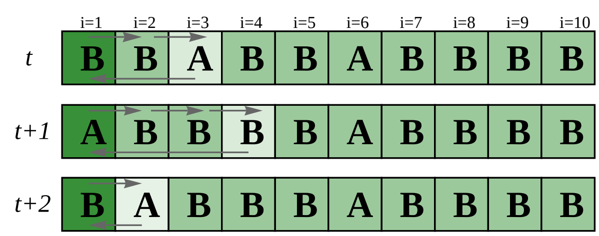

FIG. 1: Dynamics of the list. In every timestep a random position is chosen from the list and the activity sitting at the chosen position jumps to the first position and is going to be executed. The other activities are shifted to fill in the gap.

mechanism is responsible for producing the bursty behaviour because once a person starts to do an activity, that is going to have high priority for a while. We analyse the model in which the survival function $Q_n=1-\sum_{k=1}^{n}w_k$ is power-law decaying, i.e. $Q_n=cn^{-\sigma+1}+\mathscr{O}(n^{-\sigma})$. $\mathscr{O}(n^{\omega})$ means that that term is asymptotically at most of the order of n^{ω}. In this case w_i is also power-law decaying with exponent σ. Exponentially decaying priorities would result in trapped activities, meaning

that they get executed many times in sequence. If an activity happens to leave the trap it may never be executed again (in the infinite system limit). Hence such choice-distribution cannot lead to power low decaying interevent time distribution.

The model is capable of covering many types of activities but now we only concentrate on one activity marked with A in fig.1 (e.g. watching movies). For the sake of simplicity we assume that the list contains only one item of the activity of interest.

The interevent time is equal to the recurrence time of the first element of the list. Numerical results show that the interevent time distribution is power-law decaying with an exponential cutoff. The cutoff is the consequence of reaching the end of the list from which a geometrically distributed waiting time follows: $P_{wt}(t) \sim (1-w_N)^t$. For the sake of simplicity we determine the exponent of power-law decaying region in the case of an infinitely long list where the exponential cutoff does not appear.

Solution for $N \to \infty$, ($\sigma > 1$): Let $q(n,t)$ $(n \geq 1)$ denote the probability of finding the observed element at position n after t timesteps without any recurrences up to time t. The restriction not to recur is important because this makes large jumps to the front of the list *forbidden* for the observed element. The initial condition is set as $q(n,1) = (1-w_1)\delta_{n,2}$ and time evolution is given by

$$q(1,t) \equiv 0 \tag{1}$$

$$q(n,t+1) = (1-Q_{n-1})q(n,t) + Q_{n-1}q(n-1,t) \tag{2}$$

Our aim is to determine the asymptotic behaviour of $\sum_{n=1}^{\infty} q(n,t)$. The main trick we use in analysing the recursion above is to consider the $n = const$ levels first instead of the $t = const$ levels which intuitively one would do. At every step the time coordinate gets increased while the position of the element might remain unchanged or get increased by one as well (fig.2). The path of the element in the (n,t) plane can

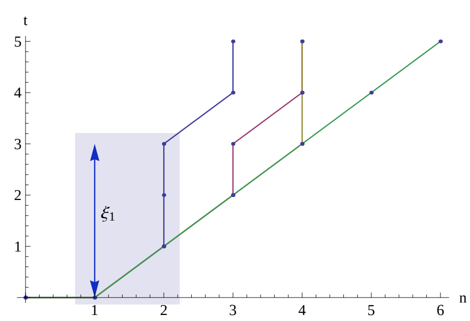

FIG. 2: Some examples for the path in the (n,t) plane. A typical section in the path corresponding to the ξ_k random variable is emphasized by a rectangle.

be divided into sections that start with a step on the bias and are followed by some steps upwards (this can be zero). These sections can be characterised by their height-difference which can take values from $1,2,\ldots$. These height differences are independent and (optimistic) geometrically distributed with parameter depending on the position. Let ξ_k be independent geometrically distributed random variables with parameter Q_k, i.e. $\mathbb{P}(\xi_i = \tau) = (1-Q_i)^{\tau-1}Q_i$. With n fixed $q(n,t)$ is the probability that we find the element at the n^{th} position at time t. This corresponds to paths with $n-1$ steps to the right (started from position 1 at time 0) and $q(n,t)$ is the distribution mass function of the sum of the random heights.

$$q(n,t) = \mathbb{P}\left(\sum_{i=1}^{n-1} \xi_i = t\right) \tag{3}$$

Though the random variables ξ_k are not identically distributed – by somewhat tedious but otherwise standard Fourier analytic methods (analysis of characteristic functions) – one can show that the central limit theorem holds for this situation. In this approximation $q(n,t)$ is Gaussian in variable t with mean $\sum_{i=1}^{n-1} \mathbb{E}(\xi_i)$ and variance $\sum_{i=1}^{n-1} \mathbb{D}^2(\xi_i)$. Using integral approximation to evaluate the sums and keeping only the highest order terms in n yields:

$$q(n,t) \overset{CLT}{\approx} \frac{c\sqrt{2\sigma-1}}{\sqrt{2\pi}n^{\sigma-1/2}} \exp\left\{-\frac{c^2(2\sigma-1)}{2}\frac{\left(t-\frac{n^\sigma}{\sigma}\right)^2}{n^{2\sigma-1}}\right\} \tag{4}$$

This formula shows that the probability of finding an element at position n at time t is centered on the $t = \frac{n^\sigma}{\sigma}$ curve which is in agreement with the numerical results (fig.3). We are interested in the sum of $q(n,t)$ in variable n because that gives the survival function of the interevent time distribution. We approximate this sum by integral and we apply the following substitution (with new variable r):

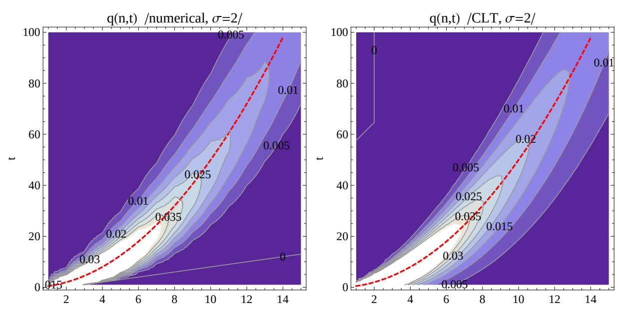

FIG. 3: Contour plots of $q(n,t)$. The image on the left is a numerical result, the right plot shows the CLT approximation. The dashed curve is $t = \frac{n^\sigma}{\sigma}$.

$$n = (c\sigma t)^{1/\sigma} - r(c\sigma t)^{1/(2\sigma)} \tag{5}$$

For $\gamma > 0$ (here γ may refer to: $\sigma - 1/2, \sigma, 2\sigma - 1$):

$$n^\gamma = (c\sigma t)^{\frac{\gamma}{\sigma}} - \gamma(c\sigma t)^{\frac{2\gamma-1}{2\sigma}}r + o(t^{\frac{2\gamma-1}{2\sigma}}), \tag{6}$$

where $o(n^\omega)$ means that that term is asymptotically of strictly smaller order than n^ω. From equations (4-(6)) it follows that

$$\int_0^\infty q(n,t)dn \approx c^{\frac{1}{\sigma}}(\sigma t)^{\frac{1}{\sigma}-1} + o(t^{1-\frac{1}{\sigma}}). \tag{7}$$

Differentiating this equation gives the first main result of our letter, $P_{ie}(t) \sim t^{-\beta}$ with $\beta = 2 - \frac{1}{\sigma}$.

Another characteristic property of the time series is autocorrelation function. Let $X(t)$ denote the indicator variable of the observed activity: $X(t) = 1$ if the observed activity is at the first position of the list at time t (i.e. an event happened) and $X(t) = 0$ otherwise. By definition the autocorrelation function:

$$\mathscr{A}(t) = \frac{\mathbb{E}[X(0)X(t)] - \langle \mathbb{E}[X(t)] \rangle_t^2}{\langle \mathbb{E}[X(t)] \rangle_t - \langle \mathbb{E}[X(t)] \rangle_t^2}. \quad (8)$$

The stationary solution of the Markov-chain defined by the model is uniform, hence $\mathbb{E}[X(t)] = \frac{N-1}{N}$ and

$$\mathscr{A}(t) = \frac{\mathbb{P}(X(t) = 1 | X(0) = 1) - \frac{N-1}{N}}{1 - \frac{N-1}{N}}. \quad (9)$$

The probability of finding the activity at the first position can be calculated numerically by successive application of the Markov-chain transition matrix. Numerical computations show that the autocorrelation function is power-law decaying with an exponential cutoff (fig.4). Given σ, the autocorrelation functions for various list sizes can be rescaled to collapse into a single curve (fig.4, inset). This property can be written

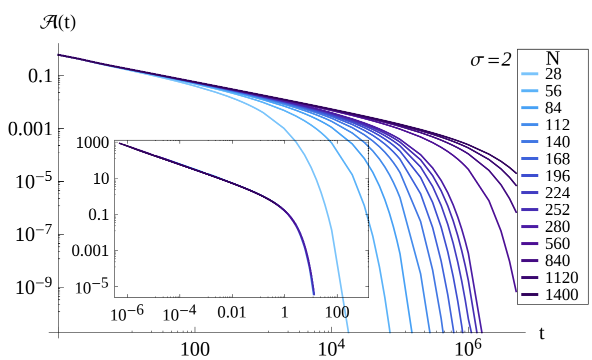

FIG. 4: Autocorrelation function of the model with various list sizes. Inset: the curves corresponding to different Ns collapse into a single curve if $N^{\delta}A(t)$ is plotted as a function of t/N^{γ}.

in a mathematical way:

$$\mathscr{A}(t, N, \sigma) = N^{-\delta} f_{\sigma}\left(\frac{t}{N^{\gamma}}\right) = t^{-\frac{\delta}{\gamma}} \tilde{f}_{\sigma}\left(\frac{t}{N^{\gamma}}\right) \quad (10)$$

The exponents used for rescaling the autocorrelation functions are listed in tab.I. indicating that in the $N \to \infty$ limit the autocorrelation function is power-law decaying with exponent $\alpha = \frac{1}{\sigma}$ for $\sigma \geq 2$. With the exact $\beta = 2 - \frac{1}{\sigma}$ result this can be combined into a scaling law: $\alpha + \beta = 2$.

σ	0.5	0.8	1	1.5	1.8	2	3	4	5
γ	1.07	1.15	1.17	1.57	1.8	2	3	4	5
δ	1	1	1	1	1	1	1	1	1

TABLE I: Numerical results for data collapse in fig.4: scaling parameters γ and δ for various values of σ.

Proof of the scaling law: The essential properties of the model for the scaling relation are that the interevent times are independent and power-law decaying. Let T denote the set of recurrence times and let τ be an interevent time. The autocorrelation function can be written in the following form:

$$\mathscr{A}(t) = \frac{\mathbb{P}(t \in T) - \frac{1}{\mathbb{E}[\tau]}}{1 - \frac{1}{\mathbb{E}[\tau]}} \quad (11)$$

which simplifies to $\mathscr{A}(t) = \mathbb{P}(t \in T)$ if $\beta \leq 2$. The Laplace transform of the autocorrelation function can be expressed by the Laplace transform of the interevent time distribution:

$$g(\lambda) \equiv \sum_{t=0}^{\infty} \mathscr{A}(t) e^{-\lambda t} = \left(1 - \mathbb{E}\left[e^{-\lambda \tau}\right]\right)^{-1} \quad (12)$$

Tauberian and Abelian theorems connect the asymptotics of a function with the asymptotics of its Laplace transform [19]. Applying Abelian theorem to the right side of the last equation results in $g(\lambda) \sim \lambda^{1-\beta}$. Then applying the Tauberian theorem yields $\mathscr{A}(t) \sim t^{\beta-2}$ or

$$\alpha + \beta = 2 \quad (13)$$

To be precise we had to use an extended version of the Abelian theorem which can be derived from the original theorem using integration by parts. With similar train of thought the scaling law can be extended to the $2 < \beta < 3$ region where $\beta - \alpha = 2$ holds. These results are in agreement with [16, 18].

The independence of interevent times was important in the proof of the scaling law. As a counterexample we constructed a long-range dependent set of interevent times by Metropolis-algorithm. The base of the algorithm is constructing a Markov chain on the integers that has power-law decaying stationary distribution $P(x) \sim x^{-\beta}$. The algorithm uses a proposal density (transition rate) $Q(x', x_n)$ which generates a proposal sample from the current value of the interevent time. This sample is accepted for the next value with probability

$$\alpha(x', x_n) = \min\left\{\frac{Q(x_n, x')P(x')}{Q(x', x_n)P(x_n)}, 1\right\}, \quad (14)$$

otherwise the previous value is repeated. To generate a power-law distributed sample with long-range dependency the mixing of the Markov chain should be slow, i.e. the gap in the spectrum of the Markov chain should vanish. In this order we allow only small differences between consecutive interevent times: $Q(x', x_n) = \frac{1}{2D}\mathbb{I}(x_n - D, x_n + D)$ or equivalently x' as a random variable is discrete uniformly distributed: $x' \sim DU[x_n - D, x_n + D]$. The simulation results (fig.5) show that the autocorrelation function decays slower than it would be assumed from the scaling law (5). This means that *power-law decaying autocorrelation function signs long-range dependencies only if the exponents violate the scaling relation* (13).

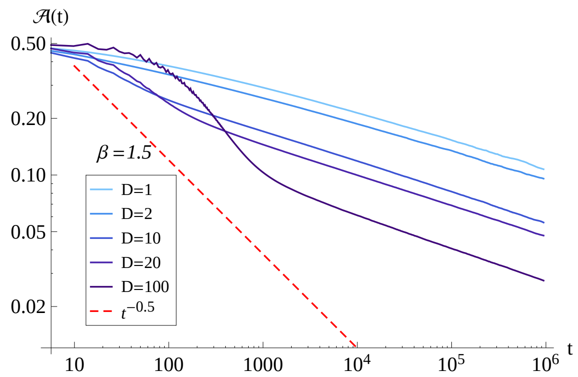

FIG. 5: Simulation results of the Metropolis algorithm. The exponent of the interevent time distribution is set to $\beta = 1.5$ and parameter D of the proposal density is varied. The dashed line shows the decay corresponding to the scaling law of independent processes.

A trivial extension of the model could be putting more items of the observed activity into the list. Then this parameter would tune the frequency of doing the activity. In this case the interevent times become dependent and simulation results show that the interevent time distribution decays faster than power-law but slower than exponential. It is easy to prove that in spite of this the autocorrelation function (9) is independent of the number of the observed activity and remains power-law decaying.

Different functions for the choice-probability w_i will result in different interevent time distributions, however, the method we introduced using the central limit theorem is general and can be applied in many cases.

The model may be useful not only for human dynamics but also for the theory of card shuffling. We can define the time reversed version of the model in which we choose a position from the same distribution as in the original model and we move the first element of the list to the random position. This model has similar properties to the original model, e.g. this model has the same interevent time distribution and autocorrelation function. If we think of the list as a *deck of cards*, then the time reversed model is a generalisation of the *top-in-at-random shuffle* method. The latter is a primitive model of shuffling cards: the top card is removed and inserted into the deck at a uniformly distributed random position [20].

In a proper model one should take into consideration the dependency among the interevent times besides the interevent time distribution. In human dynamics the latter is slowly decaying and as a consequence of this the autocorrelation function of the interevent times is not well-defined (i.e. the second moment does not exist). However, the autocorrelation function of the time series exists and it might be a good measure for long range dependency among the interevent times. Time series of messages sent by an individual in an online community are reported to be not more correlated than an independent process with the same interevent time distribution [16]. Similarly, the exponents in the neuron-firing sequences approximately satisfy the scaling law ($\alpha = 1.0$, $\beta = 1.1$ [13]). For these systems the model we studied might be applicable. However, this is not always the case. For example, the autocorrelation function of the e-mail sequence decays slower ($\alpha = 0.75$ [13]) than it should do estimated from the scaling law. This indicates long range dependency in the time series in addition to the power-law decaying interevent time distribution. In this case a dependent model should be applied, for example a model based on Metropolis algorithm that is similar to the one we studied as a counterexample to the scaling law.

In real networks interactions between individuals have to be taken into account to reproduce some social phenomena, e.g. temporal motifs [21] observed in a mobile call dataset. Interactions could be incorporated in this model by allowing the actual activity of an individual to modify the priority list of some of his/her neighbour. If this effect is rare and can be considered as a perturbation, our results for the dynamics of the list could be a starting point for further calculations covering for example information flow in a network.

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